An ensemble related to discrete orthogonal polynomials and its application to tilings of a half-hexagon.

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Abstract

We discuss asymptotic properties of a family of discrete probability measures which may be used to model particle configurations with a wall on a set of discrete nodes. The correlations are shown to be determinantal and are expressed in terms of discrete orthogonal polynomials. As an application we study random tilings of the half-hexagon or, equivalently, configurations of non-intersecting lattice paths above a wall, so called water melons.

1 Introduction

In this article we are concerned with discrete probability distributions which are closely related to discrete orthogonal polynomial ensembles. Given a positive weight function w_N defined on a set of nodes $X_N = \{x_{N,0} < \ldots < x_{N,N-1}\}$ contained in an interval $[a, b]$, the probability distribution on the set of k-tuples $(x_1, \ldots, x_k) \in X_N^k$ with $x_1 < x_2 < \ldots < x_k$ $(k \leq N)$, given by

$$p^{(N,k)}(x_1, x_2, \ldots, x_k) = \frac{1}{Z_k} \prod_{i=1}^{k} w_N(x_i) \prod_{1 \leq i < j \leq k} (x_j - x_i)^2 \tag{1.1}$$

is called *discrete orthogonal polynomial ensemble*. Here Z_k is a normalisation constant. We refer to this ensemble as $DOPE(N, k)$. $p^{(N,k)}(x_1, \ldots, x_k)$ can be viewed as the probability of finding a configuration of k particles located at the sites $x_1, \ldots, x_k \in X_N$. Continuous siblings of these ensembles describe the joint distribution of eigenvalues of a random matrix from the Gaussian unitary ensemble [5, 10], or the joint distribution of k non-intersecting Brownian bridges, see [6].

The results of this paper deal with a related ensemble involving an *even* weight function w_{2N} (i.e. $w_{2N}(x) = w_{2N}(-x)$) which is defined on a symmetric set of nodes $X_{2N} = Y_N \cup -Y_N \subseteq [-b, b]$, where $Y_N = \{x_{2N,N}, \ldots, x_{2N,2N-1}\} \subseteq [0, b]$. We define $DOPE^{\mathrm{sym}}(N, k)$ to be the ensemble of k-tuples $(x_1, \ldots, x_k) \in Y_N^k$, $x_1 < \ldots < x_k$, equipped with the probability distribution

$$p_{\mathrm{sym}}^{(N,k)}(x_1, x_2, \ldots, x_k) = \frac{1}{Z_k^{\mathrm{sym}}} \prod_{i=1}^{k} x_i^2 w_{2N}(x_i) \prod_{1 \le i < j \le k} \left(x_j^2 - x_i^2\right)^2, \tag{1.2}$$

where Z_k^{sym} is a normalisation constant. This ensemble may serve as a model for particle configurations in the presence of an impenetrable wall, as e.g. in the half-hexagon tilings discussed below. These in turn are studied in connection with an ensemble of random matrices in [4].

In Section 2 it is shown that the ensembles $DOPE^{\mathrm{sym}}(N, k)$ and $DOPE(2N, 2k)$ with the even weight function w_{2N} are closely related via their correlation functions. In the monograph [1] asymptotic results on discrete orthogonal polynomial ensembles are established for the limit $k, N \to \infty$ with $k \sim cN$. These are applied to prove similar results for the $DOPE^{\mathrm{sym}}(N, k)$ ensemble.

Remark: Throughout the paper, when dealing with the ensemble $DOPE(2N, 2k)$ we assume that w_{2N} is an even function and the set of nodes X_{2N} is of the above symmetric form.

2 Correlation functions

We first summarise some facts on the $DOPE(N, k)$ ensembles (1.1), the most important of which is the determinantal structure of the correlation functions. It involves the eponymous orthogonal polynomials. A scalar product on the set of complex valued functions on X_N is associated to the weight function w_N via

$$(f, g) \mapsto \sum_{i=0}^{N-1} w_N\left(x_{N,i}\right) f\left(x_{N,i}\right) \overline{g\left(x_{N,i}\right)}. \tag{2.1}$$

By applying the Gram-Schmidt procedure to the sequence of monomials $1, x, \ldots x^{N-1}$ we obtain a family of *orthonormal polynomials* $p_{N,0}, \ldots, p_{N,N-1}$, i.e. the degree of $p_{N,j}$ is equal to j and the relation

$$\sum_{i=0}^{N-1} w_N\left(x_{N,i}\right) p_{N,k}\left(x_{N,i}\right) \overline{p_{N,l}\left(x_{N,i}\right)} = \delta_{kl}$$

holds. Note that since the nodes, the weights and the coefficients of the $p_{N,k}$ are real, we can omit complex conjugation. Furthermore the leading coefficient $\gamma_{N,k}$ of $p_{N,k}$ is assumed to be positive. Denote by $\pi_{N,k} := \gamma_{N,k}^{-1} p_{N,k}$ the kth monic orthogonal polynomial. The m-point correlation functions $R_{N,k}^m(x_1, \ldots, x_m)$ describe the probability that a configuration

of k particles in $DOPE(N,k)$ contains particles at each of the m sites $x_1,\ldots,x_m$ ($m \leq k$). In particular the one-point correlation function $R^1_{N,k}(x)$ equals the probability of finding a particle at x. We have

$$\begin{aligned} R^m_{N,k}(x_1,\ldots,x_m) &= \mathbb{P}(\text{particles at each of the sites } x_1,\ldots,x_m) \\ &= \det\left(K_{N,k}\left(x_i,x_j\right)\right)_{i,j=1,\ldots,m}, \end{aligned} \tag{2.2}$$

where for $x,y \in X_N$ the correlation kernel $K_{N,k}(x,y)$ is given by

$$\begin{aligned} K_{N,k}(x,y) &= \sqrt{w_N(x)w_N(y)}\sum_{n=0}^{k-1} p_{N,n}(x)p_{N,n}(y) \\ &= \sqrt{w_N(x)w_N(y)}\cdot\frac{\gamma_{N,k-1}}{\gamma_{N,k}}\cdot\frac{p_{N,k}(x)p_{N,k-1}(y)-p_{N,k}(y)p_{N,k-1}(x)}{x-y} \end{aligned} \tag{2.3}$$

if $x \neq y$, and otherwise

$$K_{N,k}(x,x) = w_N(x)\cdot\frac{\gamma_{N,k-1}}{\gamma_{N,k}}\cdot\left(p'_{N,k}(x)p_{N,k-1}(x)-p'_{N,k-1}(x)p_{N,k}(x)\right). \tag{2.4}$$

The second "=" in equation (2.3) is known as Christoffel-Darboux formula. The derivations of both the particular form of the correlation functions and the summation formula are fundamental calculations in random matrix theory, see [5, Ch. 4] or [10, Ch. 5].

We can also obtain determinantal representations for the m-point correlation functions of the $DOPE^{\text{sym}}(N,k)$ ensembles (1.2). To this end we need monic polynomials $q_j(z)$ of degree j, $j=0,\ldots,N-1$ with the property

$$\sum_{x\in Y_N} x^2 w_{2N}(x) q_i\left(x^2\right) q_j\left(x^2\right) = \frac{\delta_{ij}}{\epsilon_i^2}.$$

Once these are at hand, we can mimic the standard random matrix computations [5, Ch. 4] and find a determinantal representation of the m-point correlation function with kernel

$$K^{\text{sym}}_{N,k}(x,y) = \sqrt{x^2 w_{2N}(x)}\sqrt{y^2 w_{2N}(y)}\sum_{n=0}^{k-1}\epsilon_n^2 q_n\left(x\right) q_n\left(y\right).$$

Peter Forrester pointed out how these polynomials q_j can be obtained. Consider the monic polynomials orthogonal with respect to the even weight function w_{2N} on the set of nodes $X_{2N} = Y_N \cup -Y_N$,

$$\sum_{x\in X_{2N}} w_{2N}(x)\pi_{2N,i}\left(x\right)\pi_{2N,j}\left(x\right) = \frac{1}{\gamma^2_{2N,i}}\delta_{ij}.$$

Since w_{2N} is even, it follows easily from the Gram-Schmidt procedure that $\pi_{2N,2j}(x)$ is even and $\pi_{2N,2j+1}(x)$ is odd. In particular we have $\pi_{2N,2j+1}(x) = xq_j\left(x^2\right)$ for a monic polynomial

q_j of degree j. The q_j satisfy

$$
\begin{aligned}
&\sum_{x\in Y_N} x^2 w_N(x) q_i\left(x^2\right) q_j\left(x^2\right) \\
=&\sum_{x\in Y_N} w_N(x)\pi_{2N,2i+1}\left(x\right)\pi_{2N,2j+1}\left(x\right) \\
=&\frac{1}{2}\sum_{x\in X_{2N}} w_N(x)\pi_{2N,2i+1}\left(x\right)\pi_{2N,2j+1}\left(x\right)=\frac{1}{2\gamma_{2N,2i+1}^2}\delta_{ij}
\end{aligned} \tag{2.5}
$$

and hence are the sought for polynomials. For $x, y \in Y_N$ the correlation kernel for the ensemble (1.2) can be written as

$$
\begin{aligned}
K_{N,k}^{\mathrm{sym}}(x,y) =&\sqrt{x^2 w_{2N}(x)}\sqrt{y^2 w_{2N}(y)}\sum_{n=0}^{k-1} 2\gamma_{2N,2n+1}^2 q_n\left(x\right) q_n\left(y\right) \\
=&\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\sum_{n=0}^{k-1} 2\gamma_{2N,2n+1}^2 \pi_{2N,2n+1}(x)\pi_{2N,2n+1}(y) \\
=&2\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\sum_{n=0}^{k-1} p_{2N,2n+1}(x)p_{2N,2n+1}(y) \\
=&\sqrt{w_{2N}(x)}\sqrt{w_{2N}(y)}\left[\sum_{n=0}^{2k-1} p_{2N,n}(x)p_{2N,n}(y)-\sum_{n=0}^{2k-1} p_{2N,n}(x)p_{2N,n}(-y)\right] \\
=&K_{2N,2k}(x,y)-K_{2N,2k}(x,-y).
\end{aligned} \tag{2.6}
$$

In the following we will show that for $x, y \in Y_N$ and $x, y > \varepsilon > 0$ the summand $K_{2N,2k}(x,-y)$ tends to zero in the considered limit, and hence the correlation kernels $K_{2N,2k}$ and $K_{N,k}^{\mathrm{sym}}$ asymptotically coincide. For nodes which are in $O(1/N)$ distance to 0, this is in general not the case.

Remark. *i)* The $DOPE^{sym}(N,k)$ ensembles can also be defined with a weight w_{2N+1} on a symmetric set of nodes $X_{2N+1} = -Y_N \cup \{0\} \cup Y_N$. Configurations containing 0 occur with probability 0 and the calculations of equation (2.5) also apply in this situation, leading to a kernel $K_{N,k}^{\mathrm{sym}} = K_{2N+1,2k}(x,y) - K_{2N+1,2k}(x,-y)$. For notational convenience, we discuss the case of an even number of nodes.
ii) A similar relation is present between ensembles of non-intersecting Brownian motions and excursions [13], the continuous analogues of the non-intersecting lattice path models corresponding to random tilings of the full and the half-hexagon, respectively.

3 Discrete orthogonal polynomials

In order to state our asymptotic results, we review some terminology and results from the monograph [1]. In applications as the arctic circle phenomenon, the asymptotic behaviour

of $p_{N,k}(z)$ and $K_{N,k}$ as N and k *simultaneously* tend to infinity plays a crucial role. The results of [1] are obtained under some technical assumptions [1, Section 1.2] on the weight, nodes and number of particles. These are in particular fulfilled in the situation of the (half-) hexagon tilings.

3.1 Basic assumptions

The nodes

We assume the existence of a *node density function* ρ^0, which is real-analytic in a complex neighbourhood of $[a,b]$, strictly positive in $[a,b]$ and satisfies a normalisation condition and a certain quantisation rule, namely

$$\int_a^b \rho^0(x)\mathrm{d}x = 1 \text{ and } \int_a^{x_{N,n}} \rho^0(x)\mathrm{d}x = \frac{2n+1}{2N}, \; n = 0, \ldots, N-1. \tag{3.1}$$

The weight function

We assume that we can write the weight function in the form

$$w_N\left(x_{N,n}\right) = (-1)^{N-n-1} e^{-NV_N\left(x_{N,n}\right)} \prod_{\substack{m=0 \\ n\neq m}}^{N-1} \left(x_{N,n} - x_{N,m}\right)^{-1}, \tag{3.2}$$

where $V_N(x)$ is a real-analytic function defined in a neighbourhood G of $[a,b]$. Furthermore

$$V_N(x) = V(x) + \frac{\eta(x)}{N}, \tag{3.3}$$

where $V(x)$ is a fixed real-analytic potential function independent of N and

$$\limsup_{N\to\infty} \sup_{z\in G} |\eta(z)| < \infty.$$

As opposed to $V(x)$, the correction $\eta(x)$ may depend on N.

The number of particles

The number of particles k (at the same time the degree of the involved polynomials) and the number of nodes N are related by

$$k = cN + \kappa, \tag{3.4}$$

where $c \in (0,1)$ and κ remains bounded as $N \longrightarrow \infty$.

Further assumptions are difficult to express explicitly in terms of the nodes and the weight and postponed to Section 3.3.

3.2 The associated equilibrium energy problem

The asymptotic behaviour of $\pi_{N,k}$ is closely related to the asymptotic distribution of zeroes in the interval $[a,b]$, wich in turn can be expressed in terms of quantities arising in a related constrained variational problem [9]. Define a real-analytic function φ by

$$\varphi(x) := V(x) + \int_a^b \log|x-y|\mathrm{d}y. \tag{3.5}$$

Note that according to the representation (3.2) $w_N(x)$ is asymptotically equal to

$$w_N(x) \sim e^{-N\varphi(x)-\eta(x)}. \tag{3.6}$$

Recall that $\pi_{N,k}$ minimises the scalar product (2.1) among all monic polynomials of degree k. This relates the asymptotic distribution of zeroes of $\pi_{N,k}$ to (the variational derivative of) the quadratic functional of Borel measures on $[a,b]$ defined by

$$E_c[\mu,V] := E_c[\mu] := c\int_a^b\int_a^b \log\frac{1}{|x-y|}\mathrm{d}\mu(x)\mathrm{d}\mu(y) + \int_a^b \varphi(x)\mathrm{d}\mu(x). \tag{3.7}$$

We are looking for a measure $\mu^c_{\min}$ which minimises $E_c[\mu]$ subject to the normalisation condition

$$\int_a^b \mathrm{d}\mu(x) = 1 \tag{3.8}$$

and the upper and lower constraints

$$0 \le \int_{x\in B}\mathrm{d}\mu(x) \le \frac{1}{c}\int_{x\in B}\rho^0(x)\mathrm{d}x \text{ for every a Borel set } B \text{ in } [a,b]. \tag{3.9}$$

We refer to $\mu^c_{\min}$ as the *equilibrium measure.* The constraints are due to the fact that all zeroes of $\pi_{N,k}$ are contained in the interval $[x_{N,0}, x_{N,N-1}]$ and a closed interval $[x_{N,n}, x_{N,n+1}]$ between two consecutive nodes contains at most one zero of $\pi_{N,k}$. Note that minimising $E_c[\mu]$ simply subject to the normalisation condition (3.8) is formally like seeking a critical point of

$$F_c[\mu] = E_c[\mu] - l_c\int_a^b \mathrm{d}\mu(x)$$

with a Lagrange multiplier $l_c := l_c[V]$. When $\mu = \mu^c_{\min}$, l_c is a real constant. $\mu^c_{\min}$ is known to be unique and it has a piecewise analytic density $d\mu^c_{\min}(x)/dx$. Points of non-analyticity are finite in number and do not occur in points where the upper and lower constraints $d\mu^c_{\min}(x)/dx > 0$ and $d\mu^c_{\min}(x)/dx < 1/c$ hold strictly and simultaneously.

3.3 The equilibrium measure: related quantities and further assumptions:

The constraints give rise to the following

Definition 3.1. *A* band *is a maximal open subinterval of* $[a,b]$ *where* $\mu^c_{\min}$ *is a measure with a real-analytic density* $d\mu^c_{\min}(x)/dx$ *which satisfies* $0 < d\mu^c_{\min}(x)/dx < \rho^0(x)/c$*. A* void *is a maximal open subinterval of* $[a,b]$ *in which* $d\mu^c_{\min}(x)/dx \equiv 0$*, i.e. meets the* lower *constraint. A* saturated region *is a maximal open subinterval of* $[a,b]$ *in which* $d\mu^c_{\min}(x)/dx \equiv 1/c$ *and hence meets the* upper *constraint. If no stress is put on the active constraint, voids and saturated regions are referred to as* gaps*.*

As announced in Section 3.1 we make some further assumptions on the weight and the nodes which are expressed in terms of the equilibrium measure, cf. [1, Section 2.1.2]. Let $\mathcal{F}$ be the closed set of points $x \in [a,b]$, where $d\mu^c_{\min}(x)/dx = \rho^0(x)/c$ or $d\mu^c_{\min}(x)/dx = 0$, i.e. one of the constraints (3.9) is active. We assume that the connected components of $\mathcal{F}$ have non-empty interior and $a, b \in \mathcal{F}$. Furthermore we make the following assumptions on the behaviour of $d\mu^c_{\min}(x)/dx$ at endpoints of bands. Let z_0 be a band end point. If the gap at z_0 is a void, then

$$\lim_{x\to z_0,\, x\in I} \frac{1}{\sqrt{|x-z_0|}} \frac{d\mu^c_{\min}}{dx}(x) = K, \quad \text{with } 0 < K < \infty.$$

Similarly, if the gap at z_0 is a saturated region, we suppose that

$$\lim_{x\to z_0,\, x\in I} \frac{1}{\sqrt{|x-z_0|}} \left(\frac{\rho^0(x)}{c} - \frac{d\mu^c_{\min}}{dx}(x) \right) = K, \quad \text{with } 0 < K < \infty.$$

So the constraints are met like a square root.

Remark. The one-point correlation function in the hexagonal tiling problem is shown to converge pointwise to the density of the corresponding equilibrium measure in [1, Theorem 3.12], see also Theorems 4.1, 4.2 and 4.3. In the (half-) hexagon tilings there is a single band corresponding to the intersection of a vertical line with the temperate zone, the surrounding gaps to the intersection of the line with the arctic regions.

Finally we define the quantities involved in the asymptotic expressions for $\pi_{N,k}$ in a gap Γ and the band I, cf. [1, Sect. 2.1.4]. The variational derivative of $E_c[\mu]$ evaluated at $\mu = \mu^c_{\min}$ is equal to

$$\frac{\delta E_c}{\delta\mu}(x) := \left.\frac{\delta E_c[\mu, V]}{\delta\mu}\right|_{\mu=\mu^c_{\min}} (x) = -2c \int_a^b \log|x-y| \mathrm{d}\mu^c_{\min}(x) + \varphi(x).$$

We have

$$\frac{\delta E_c}{\delta\mu}(x) - l_c \begin{cases} > 0 & \text{if } x \text{ is in a void,} \\ \equiv 0 & \text{if } x \text{ is in a band,} \\ < 0 & \text{if } x \text{ is in a saturated region.} \end{cases} \tag{3.10}$$

The function $\frac{\delta E_c}{\delta\mu} - l_c$ defined in a gap Γ extends analytically from the interior. Furthermore, we denote by $\overline{L}^{\Gamma}_c(z)$ and $\overline{L}^{I}_c$ the function

$$c\int_a^b \log|z-x| \mathrm{d}\mu^c_{\min}(x),$$

defined for z, in a gap Γ or a band I, respectively. $\overline{L}_c^{\Gamma}(z)$ is analytic in z if $\Re z \in \Gamma$ and $\Im z$ sufficiently small. $\overline{L}_c^{I}$ has an analytic continuation to a neighbourhood of $\overline{I}$.

4 Statement of asymptotic results

We have now introduced all the notation required to state our main results. The following results are verbatim copies of the corresponding results for the $DOPE(2N, 2k)$ ensemble in [1], except for the "close to zero" assertions in Theorems 4.1 and 4.5.

The first results treat the band case and involve the *sine kernel*

$$S(\xi, \eta) := \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)}. \tag{4.1}$$

Extend S to the diagonal by setting $S(\xi, \xi) := 1$. Correlations for nodes close to 0 are expressed in terms of the kernel given by

$$S^0(\xi, \eta) := \frac{\sin(\pi(\xi - \eta))}{\pi(\xi - \eta)} - \frac{\sin(\pi(\xi + \eta))}{\pi(\xi + \eta)}.$$

Theorem 4.1 (Theorem 3.1 in [1]). *Denote by $R_m^{(N,k)}(x_1, \ldots, x_m)$ the m-point correlation function for the ensemble $DOPE^{\mathrm{sym}}(N, k)$ with an even weight w_{2N}. Fix $x \neq 0$ in the interior of a band $I := (\alpha, \beta)$ and let*

$$\delta(x) := \left[c \frac{d\mu_{\min}^c}{dx}(x) \right]^{-1}. \tag{4.2}$$

Let $\xi_N^{(1)}, \ldots, \xi_N^{(m)}$ belong to a fixed bounded set $D \subset \mathbb{R}$ in such a way that the points defined by

$$x_j := x + \xi_N^{(j)} \frac{\delta(x)}{N}, \ j = 1, \ldots, m$$

are all nodes in X_N. Consequently, $x_j \to x$ as $N \to \infty$. Then there is a constant $C_{D,m}$ depending on D and m such that for sufficiently large N we have

$$\max_{\xi_N^{(1)}, \ldots, \xi_N^{(m)} \in D} \left| R_m^{(N,k)}(x_1, \ldots, x_m) - \left(\frac{1}{\delta(x)} \right)^m \det \left(S\left(\xi_N^{(i)}, \xi_N^{(j)} \right) \right)_{1 \le i,j \le m} \right| \le \frac{C_{D,m}}{N}.$$

If $(-\beta, \beta)$ is a band of $d\mu_{\min}^c(x)/dx$, $x = 0$ and the nodes are

$$x_j := \xi_N^{(j)} \frac{\delta(0)}{N}, \ \xi_N^{(j)} > 0, \ j = 1, \ldots, m,$$

then there is a constant $\tilde{C}_{D,m}$ such that for sufficiently large N

$$\max_{\xi_N^{(1)}, \ldots, \xi_N^{(m)} \in D} \left| R_m^{(N,k)}(x_1, \ldots, x_m) - \left(\frac{1}{\delta(0)} \right)^m \det \left(S^0\left(\xi_N^{(i)}, \xi_N^{(j)} \right) \right)_{1 \le i,j \le m} \right| \le \frac{\tilde{C}_{D,m}}{N}.$$

Theorem 4.2 (Theorems 3.3 and 3.5 in [1]). *For a fixed closed interval F in a gap Γ there are constants K_F and $C_{F,m}$, such that for sufficiently large N and nodes $x_1, \ldots, x_m \in F \cap Y_N$*

$$\max_{x_1,\ldots,x_m \in F} \left| R_m^{(N,k)}(x_1, \ldots, x_m) \right| \leq C_{F,m} \frac{e^{-mK_F N}}{N^m}$$

holds if Γ is a void. If Γ is a saturated region, we have the estimate

$$\max_{x_1,\ldots,x_m \in F} \left| R_m^{(N,k)}(x_1, \ldots, x_m) - 1 \right| \leq C_{F,m} \frac{e^{-K_F N}}{N}.$$

The result for nodes close to a band end point is expressed in terms of the *Airy kernel*

$$A(\xi, \eta) := \frac{\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta) - \mathrm{Ai}'(\xi)\mathrm{Ai}(\eta)}{\xi - \eta}. \tag{4.3}$$

Theorem 4.3 (Theorems 3.7 and 3.8 in [1]). *For each fixed $M > 0$, each integer m and a right band end point β separating I from a void, there is a constant $G_\beta^m(M)$ such that for sufficiently large N*

$$\max \left| R_m^{(N,k)}(x_1, \ldots, x_m) - \left[\frac{(\pi c B_\beta)^{2/3}}{(2N)^{1/3} \rho^0(\beta)} \right]^m \det \left(A(\xi_N^{(i)}, \xi_N^{(j)}) \right)_{1 \leq i,j \leq m} \right| \leq \frac{G_\beta^m(M)}{N^{(m+1)/3}},$$

where the max *is taken over nodes $x_1, \ldots, x_m \in X_N$ all satisfying*

$$\beta - M(2N)^{-2/3} < x_j < \beta - M(2N)^{-1/2},$$

the constant B_β is equal to (cf. Sect. 3.3)

$$B_\beta := \lim_{x \uparrow \beta} \frac{1}{\sqrt{\beta - x}} \frac{d\mu_{\min}^c}{dx}(x) > 0$$

and $\xi_N^{(j)} = (2N\pi c B_\beta)^{2/3} (x_j - \beta)$. If β is adjacent to a saturated region, then

$$\max \left| R_m^{(N,k)}(x_1, \ldots, x_m) - 1 + \frac{(\pi(1-c)\overline{B}_\beta)^{2/3}}{(2N)^{1/3} \rho^0(\beta)} \sum_{j=1}^m A(\xi_N^{(j)}, \xi_N^{(j)}) \right| \leq \frac{H_\beta^m(M)}{N^{2/3}},$$

where $H_\beta^m(M)$ is a constant, $\xi_N^{(j)} = \left(2N\pi(1-c)\overline{B}_\beta\right)^{2/3} (x_j - \beta)$ and $\overline{B}_\beta$ is equal to

$$\overline{B}_\beta = \lim_{x \uparrow \beta} \frac{1}{\sqrt{\beta - x}} \frac{c}{1-c} \left[\frac{1}{c} \rho^0(x) - \frac{d\mu_{\min}^c}{dx}(x) \right] > 0.$$

Remark. For $DOPE(2N, 2k)$ this result is Theorem 3.7 and 3.8 in [1]. Analogous results hold for left band end points.

Another interesting statistic concerning particle systems is the fluctuation of extremal particles, e.g. the intersection point of a vertical line with the boundary of the arctic

region in random tilings. More generally, let $B \subset X_N$ be a set of nodes and m an integer such that $0 \le m \le \min(\#B, k)$. A well studied statistic is

$$\begin{aligned} A_m^{(N,k)}(B) &:= \mathbb{P}(\text{there are exactly } m \text{ particles in } B) \\ &= \frac{1}{m!}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \det\left(\mathbb{I} - t\, K_{N,k}|_B\right)\Big|_{t=1}, \end{aligned}$$

where $K_{N,k}^{\text{sym}}$ is the operator on $\ell_2(Y_N)$ with kernel $K_{N,k}^{\text{sym}}(x,y)$ and $K_{N,k}|_B$ its restriction to $\ell_2(B)$.

Let β be the rightmost band end point and (β, b) be a void (resp. saturated region) Denote by $x_{\max}$ the position of the rightmost particle (resp. hole, i.e. unoccupied node). Since the one-point function of $DOPE(2N, 2k)$ converges pointwise to $c\, d\mu_{\min}^c(x)/dx$ and $DOPE^{\text{sym}}(N, k)$ has the same asymptotic behaviour in regions bounded away from 0, one expects $x_{\max}$ near the rightmost band edge β. In this domain the correlation kernel approximates the Airy kernel. The latter kernel is the correlation kernel of the distribution of eigenvalues of a GUE matrix at the edge of the spectrum and the fluctuations of the largest eigenvalue are governed by the *Tracy-Widom distribution* ν [12] whose distribution function is equal to

$$\nu((\infty, s]) = \det\left(\mathbb{I} - \mathcal{A}|_{[s,\infty)}\right),$$

where $\mathcal{A}|_{[s,\infty)}$ is the trace class operator on $L_2[s,\infty)$ defined by the Airy kernel. The position rightmost particle (properly scaled) in $DOPE(2N, 2k)$ is proved to be Tracy-Widom-distributed and the proof carries over verbatim to $DOPE^{\text{sym}}(N, k)$.

Theorem 4.4 (Theorem 3.9 in [1])**.** *Let the gap adjacent to (β, b) be a void. Then the position of the rightmost particle $x_{\max}$ in the ensemble $DOPE^{\text{sym}}(N, k)$ is described by the limiting distribution function*

$$\lim_{N\to\infty} \mathbb{P}\left(x_{\max} \le \beta + \frac{s}{(2\pi N c B_\beta)^{2/3}}\right) = \det\left(\mathbb{I} - \mathcal{A}|_{[s,\infty)}\right).$$

If the gap is a saturated region and $x_{\max}$ describes the position of the rightmost unoccupied node (hole), then the same relation holds with c and B_β replaced by $1-c$ and $\overline{B}_\beta$, respectively.

In $DOPE^{\text{sym}}(N, k)$ also the behaviour of the leftmost particle is of separate interest. Due to our assumptions in 3.3, 0 lies in the interior of a band or a gap, the gap case being covered by Theorem 4.2. So we assume that no constraint is active in 0. We denote by $\mathcal{S}^0$ the operator on $\ell_2(\mathbb{Z})$ given by

$$\mathcal{S}_{ij}^0 = \frac{\sin\left(\frac{1}{\delta(0)\rho^0(0)}\pi(i-j)\right)}{\pi(i-j)} - \frac{\sin\left(\frac{1}{\delta(0)\rho^0(0)}\pi(i+j)\right)}{\pi(i+j)}, \tag{4.4}$$

where $\delta(x)$ is defined in equation (4.2).

Theorem 4.5. *For the position of the leftmost particle $x_{\min}$ in the $DOPE^{sym}(N,k)$ ensemble we have the limiting distribution function*

$$\lim_{N\to\infty} \mathbb{P}\left(x_{\min} \geq \frac{s}{\rho^0(0)N}\right) = \det\left(\mathbb{I} - \mathcal{S}^0\big|_{B(s)}\right),$$

where $B(s)$ is the finite set of integers $\{0,1,\ldots,\lfloor s-1/2\rfloor\}$ and $\lfloor s-1/2\rfloor$ denotes the integer part of $s-1/2$.

5 Proofs of the asymptotic results

5.1 Asymptotics for $DOPE(N,k)$

We summarise weaker versions of some of the Theorems of [1], which suffice for the estimates we need below. The first is on leading coefficients.

Proposition 5.1 (Theorem 2.8 in [1]). *The leading coefficients $\gamma_{N,k}$ of $p_{N,k}(z)$ and $\gamma_{N,k-1}$ of $p_{N,k-1}(z)$ satisfy the asymptotic relations:*

$$\gamma_{N,k}^2 = C_{\gamma_{N,k}} e^{Nl_c+\gamma}(1+O(1/N))$$

and

$$\gamma_{N,k-1}^2 = C_{\gamma_{N,k-1}} e^{Nl_c+\gamma}(1+O(1/N))$$

where $C_{\gamma_{N,k-1}}$ and $C_{\gamma_{N,k}}$ are suitable constants and γ remains bounded as $N \longrightarrow \infty$.

Proposition 5.2 (Theorems 2.9, 2.13 and 2.15 in [1]). *In suitable neighbourhoods of the different subintervals of $[a,b]$ we have the following asymptotic representations of $\pi_{N,k}$.*

1. *Assume that J is a closed subinterval in a gap Γ. Then there is a neighbourhood K_J of J, and function $A(z)$ analytic on K_J and uniformly bounded in N, such that*

   $$\pi_{N,k}(z) = e^{N\overline{L}_c^{\Gamma}(z)(z)}\left(A(z)+O(1/N)\right)$$

   holds.

2. *Assume that F is a closed subinterval in a band I. Then there is a neighbourhood K_F of F, and a sequence of analytic functions $B_N(z)$ defined on K_J and uniformly bounded in N, such that*

   $$\pi_{N,k}(z) = e^{N\overline{L}_c^{I}(z)}\left(B_N(z)+O(1/N)\right).$$

3. *Let z_0 be a band endpoint. Then there is $r>0$ and sequence of functions $C_N(z)$ analytic in $|z-z_0|<r$ and uniformly bounded for $N\to\infty$, such that for $|z-z_0|<r$*

   $$\pi_{N,k}(z) = e^{N\overline{L}_c^{I}(z)}N^{1/6}\left(C_N(z)+O(1/N^{1/3})\right).$$

More precisely we have the following results, the first of which treats the band case and involves the sine kernel (4.1).

Lemma 5.3 (Lemma 7.13 in [1]). *Fix x in the interior of a band $I=(\alpha,\beta)$ and let*

$$\delta(x) := \left[c\frac{d\mu^c_{\min}}{dx}(x)\right]^{-1}.$$

Let ξ_N, η_N belong to a fixed bounded set $D\subset\mathbb{R}$ in such a way that the points defined by

$$w := x+\xi_N\frac{\delta(x)}{N},\ \ z := x+\eta_N\frac{\delta(x)}{N}$$

are both nodes in X_N. Consequently, $w,z\to x$ as $N\to\infty$. Then there is a constant C_D depending on D such that for sufficiently large N we have

$$\max_{\xi_N,\eta_N\in D}\left|K_{N,k}(w,z)-c\frac{d\mu^c_{\min}}{dx}(x)S\left(\xi_N,\eta_N\right)\right|\le\frac{C_D}{N}.$$

Furthermore, if $x_{N,i},\ x_{N,j}\longrightarrow x$ while $i-j$ remains fixed, we have

$$K_{N,k}\left(x_{N,i},\ x_{N,j}\right)=\mathcal{S}_{ij}(x)+O(1/N), \tag{5.1}$$

where

$$\mathcal{S}_{ij}(x)=\frac{\sin\left(\frac{c}{\rho^0(x)}\frac{d\mu^c_{\min}}{dx}(x)\cdot\pi(i-j)\right)}{\pi(i-j)}. \tag{5.2}$$

□

Lemma 5.4 (Lemma 7.16 in [1]). *Let β be the right band end point. For each fixed $M>0$ there is a constant $C_\beta(M)>0$, such that for N sufficiently large*

$$\max\left|K_{N,k}(x,y)-\left[\frac{(\pi cB_\beta)^{2/3}}{N^{1/3}}\right]A(\xi_N,\eta_N)\right|\le\frac{C_\beta(M)}{N^{2/3}}$$

holds, where the max *is taken over pairs of nodes $x,y\in X_N$ all satisfying*

$$\beta-MN^{-2/3}<x,y<\beta+MN^{-1/2},$$

the constant B_β is equal to (cf. Sect. 3.3)

$$B_\beta := \lim_{x\uparrow\beta}\frac{1}{\sqrt{\beta-x}}\frac{d\mu^c_{\min}}{dx}(x)>0$$

and $\xi_N=\left(N\pi cB_\beta\right)^{2/3}(x-\beta)$ and $\eta_N=\left(N\pi cB_\beta\right)^{2/3}(y-\beta)$.

Lemma 5.5 (cf. Lemma 7.4 in [1]). *We have the exact formula for any node $x = x_{N,n} \in X_N$*

$$\sqrt{w_N(x)} = \frac{1}{\sqrt{\pi\rho^0(x)N}} e^{-\frac{1}{2}\eta(x)} e^{-\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(x)-l_c\right)} e^{-\frac{1}{2}(Nl_c+\gamma)} e^{-N\int_a^b \log|x-y|\mathrm{d}\mu^c_{\min}(y)} T_N(x)^{\frac{1}{2}}$$

where the function

$$T_N(z) = \cos\left(\frac{2\pi N\int_z^b \rho^0(x)\mathrm{d}x}{2}\right) \frac{1}{\prod_{m=0}^{N-1}(z - x_{N,m})} e^{N\int_a^b \log|z-x|\rho^0(x)\mathrm{d}x}$$

is real analytic in (a, b) and bounded independently of N.

Proof. Plug in the definitions and compare to the representation (3.2) of w_N, this verifies the exponentials. The product in (3.2) is rewritten as

$$\prod_{\substack{m=0\\ m\neq n}}^{N-1} (x_{N,n} - x_{N,m})^{-1} = \lim_{z\to x_{N,n}} \frac{z - x_{N,n}}{\cos\left(\frac{2\pi N\int_z^b \rho^0(x)\mathrm{d}x}{2}\right)} \cdot \frac{\cos\left(\frac{2\pi N\int_z^b \rho^0(x)\mathrm{d}x}{2}\right)}{\prod_{m=0}^{N-1}(z - x_{N,m})}$$

The limit of both fractions exists. De l'Hôspital's rule applied to the first fraction yields

$$\frac{1}{\pi\rho^0(x)N\sin\left(\frac{2\pi N\int_x^b \rho^0(x)\mathrm{d}x}{2}\right)} = \frac{1}{\pi\rho^0(x)N(-1)^{N-n-1}}.$$

This completes the proof. □

Remark. $T_N(z)$ has no poles at the nodes as the zeroes in the denominator are cancelled by the cosine. Furthermore the product in the denominator is asymptotically equal to the exponential for $N \to \infty$.

Proposition 5.6 (Lemma 7.12 and 7.14 in [1]). *Denote by E the finite set $\{\alpha_0, \beta_0, \ldots, \alpha_h, \beta_h\}$ of band end points. Let F be a fixed closed subset of the interval (a, b) such that $F \cap X_N \neq \varnothing$ and $F \cap E = \varnothing$ (and hence all band end points are bounded away from F). Then, for N sufficiently large, we have for all $x, y \in F \cap X_N$ the estimate*

$$\left|(x-y)K_{N,k}(x,y)e^{\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(x)-l_c\right)}e^{\frac{1}{2}N\left(\frac{\delta E_c}{\delta\mu}(y)-l_c\right)}\right| < \frac{C}{H(N)} \tag{5.3}$$

with $H(N) = N$ and a constant C only depending on F. Furthermore, if $x, y \in X_N$ are chosen from a sufficiently small neighbourhood G of E, the estimate holds with $H(N) = N^{2/3}$ and the constant C only depending on G.

Proof. Substitute the result of Lemma 5.5 and the assertions of Proposition 5.2 into the following formula (cf. (2.3))

$$\begin{aligned}(x-y)&K_{N,k}(x,y)\\ &= \sqrt{w_N(x)w_N(y)} \cdot \gamma^2_{2N,2k-1} \cdot (\pi_{N,k}(x)\pi_{N,k-1}(y) - \pi_{N,k}(y)\pi_{N,k-1}(x))\end{aligned}$$

and estimate the uniformly bounded parts by a constant. □

Remark. Recall that $\frac{\delta E_c}{\delta \mu}(z) - l_c$ is strictly positive for z in the interior of a void and identically zero in a band. So, if at least one of x, y lies in a void, $K_{N,k}(x,y)$ is exponentially small. Moreover, under the assumptions of Lemma 5.6, $K_{N,k}(x,y)$ can be asymptotically non-zero only if $x, y \in F \cap I$ and $|x-y| = O(1/N)$ or, for x, y close to a band end point, $|x-y| = O(1/N^{2/3})$.

We can now harness the results for $DOPE(2N, 2k)$ for the ensemble $DOPE^{\text{sym}}(N,k)$. By equation (2.6) the correlation kernel for $DOPE^{\text{sym}}(N,k)$ is

$$K^{\text{sym}}_{N,k}(x,y) = K_{2N,2k}(x,y) - K_{2N,2k}(x,-y).$$

We show that $K_{2N,2k}(x,-y)$ is asymptotically neglegible and the results of the previous section also hold for $DOPE^{\text{sym}}(N,k)$.

Lemma 5.7. *Assume the situation for the $DOPE^{\text{sym}}(N,k)$ ensemble, i.e. $[a,b] = [-b,b]$ and $X_{2N} = -Y_N \cup Y_N$ for a set of nodes $Y_N = \{x_{2N,N+j},\ j = 0,\dots,N-1\}$ and the weight function w_{2N} is supposed to be even. This implies the same symmetry for the equilibrium measure $\mu^c_{\min}$. Moreover, we have the following estimates.*

1. *Let $x, y \in I \cap Y_N$ be a pair of nodes with $x, y > \delta_1$, where δ_1 is a strictly positive constant independent of N. There is a constant C depending only on δ_1 such that*
$$|K_{2N,2k}(x,-y)| < \frac{C}{N}. \tag{5.4}$$
2. *For nodes $x, y \in J \cap X_{2N}$, where J is a closed interval in a gap Γ, there are positive constants D_1, D_2 depending on J, such that*
$$|K_{2N,2k}(x,-y)| < D_1 e^{-D_2 N}.$$
3. *Let $M > 0$ and β be the right band end point. Then there is a constant C_M such that for any pair of nodes x, y with*
$$\beta - r(2N)^{-2/3} < x, y < \beta + r(2N)^{-1/2}$$
we have
$$|K_{2N,2k}(x,-y)| < \frac{C_r}{N^{2/3}}. \tag{5.5}$$

Proof. The symmetry of the weight w_{2N} implies the symmetry of V_{2N} and V (cf. equations (3.2) and (3.3)). Hence the external field $\varphi(x)$ defined in (3.5) is even. It follows that the functional $E_c[\mu]$ in (3.7) is invariant under the transformation $\mu(x) \mapsto \mu(-x)$. By uniqueness the symmetry of $\mu^c_{\min}$ follows.

The other assertions follow directly from Lemma 5.6, since in the first assertion $|x - (-y)| > 2\delta_1$ is bounded away from zero. For the second assertion recall that by equation (3.10) $\frac{\delta E_c}{\delta \mu}(x) - l_c$ is negative and bounded away from zero if x is bounded away from the band. Hence we have exponential decay in this case. For the third assertion notice that $-\beta$ is the left band end point and $|x - (-y)|$ is asymptotically equal to 2β. $\square$

Lemma 5.7 shows that, after an adjustment of the constants, the estimates for $K_{2N,2k}(x,y)$ in Lemmas 5.3 and 5.4 also hold for $K^{\mathrm{sym}}_{N,k}$. Theorem 4.1 and the "void" cases of Theorem 4.2 and Theorem 4.3 now follow by simply taking determinants. For the "saturated region" cases one has to take a detour over the dual ensemble, which describes the distribution of *hole configurations*, i.e. the probability $\overline{p}^{(N,N-k)}(y_1,\ldots,y_{N-k})$ to find the sites $y_1,\ldots,y_{N-k}$ unoccupied. Saturated regions are voids for the dual ensemble and vice versa. The proofs are carried out for $DOPE(N,k)$ in [1] and can be applied almost verbatim to $DOPE^{\mathrm{sym}}(N,k)$.

6 Example: tilings of the half-hexagon

The (Q,R,S)-*hexagon* is a hexagon with integer side lengths Q, R, S, Q, R, S and every angle equal to 120°. We study tilings thereof with 60° unit rhombi referred to as *lozenges* or simply *tiles*. The hexagon is filled without gaps and overlap, and no tile juts out beyond the boundary ("fixed boundary conditions"). The tiles occur in three different species (orientations) referred to as *up-, vertical* and *down*-tiles, see figure 1. As we are interested in a certain symmetry class we restrict to hexagons with $Q=2k$ and $R=S$. To quantify things, we fix an ON-coordinate system and look at (symmetric) tilings of the $(2k,R,R)$-hexagon with corners $(\pm\sqrt{3}R/2,\pm k)$ and $(0,\pm(k+R/2)$. The corners are numbered in counterclockwise order starting with $P_1=(-\sqrt{3}R/2,-k)$. If we focus upon the symmetry class of tilings w.r.t. the reflection in the x-axis, we can throw away the part below the x-axis and the chopped-in-half vertical tiles on the axis and obtain a tiling of the so-called (k,R)-*half-hexagon*, a model studied in [4].

Recall that tilings of the full $(2k,R,R)$-hexagon map bijectively to families of $2k$ non-intersecting paths on the triangular point lattice

$$L=\left\{\left(\frac{-\sqrt{3}}{2}R,\frac{1}{2}\right)+q\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)+r\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\ q,r\in\mathbb{Z}\right\}$$

with starting points in the set $S=S^+\cup S^-$ and end points in $E=E^+\cup E^-$, where

$$S^+=\left\{\left(-\frac{\sqrt{3}}{2}R,i+\frac{1}{2}\right),\ i=0\ldots k-1\right\},\ E^+=\left\{\left(\frac{\sqrt{3}}{2}R,i+\frac{1}{2}\right),\ i=0\ldots k-1\right\},$$

and S^- (resp. E^-) denotes the reflection of S^+ (resp. E^+) in the x-axis. Admissible steps are $(x,y)\to(x+\sqrt{3}/2,y\pm 1/2)$. To see this, connect in each up- and in each down-tile the midpoints of the vertical sides by a straight line segment (the decoration depicted in figure 1).

In the same fashion tilings of the half-hexagon are mapped to families of k non-intersecting paths with starting points in S^+ and end points in E^+ which *do not touch the x-axis*. Denote by $L_m,\ m=0,\ldots,2R$ the set of numbers such that $\{(-R+m)\sqrt{3}/2\}\times L_m$ is the intersection of the vertical line $x=(-R+m)\sqrt{3}/2$ with the lattice L and the

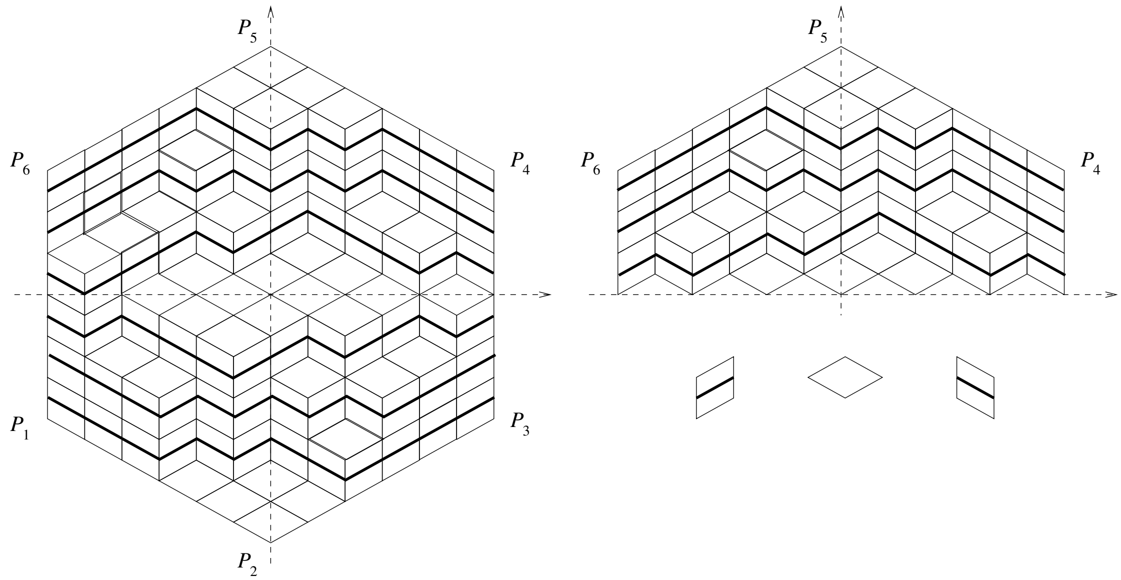

Figure 1: Tiling of a $(2k, R, R)$-hexagon (without symmetry), tiling of a (k, R)-half-hexagon, up-, vertical and down-tiles

$(2k, R, R)$-hexagon. Note that by symmetry $L_m = L_{2R-m}$ and, for $m \leq R$, $|L_m| = 2k + m$. Then $\{(-R + m)\sqrt{3}/2\} \times L_m$ is the set of possible points where a family of lattice paths corresponding to a tiling can intersect a vertical line after m steps. Denote by L_m^+ the set of positive elements of L_m.

Proposition 6.1 (Theorem 4.1 in [6], Lemma 2.2 in [4])**.** *Consider the sets of families of $2k$ non-intersecting lattice paths with starting points in S and end points in E (tilings of the $(2k, R, R)$-hexagon) and of families of k such lattice paths with starting points in S^+ and end points in E^+ not touching the x-axis ((k, R)-half-hexagon) to be equipped with their respective uniform distributions.*

1. *Let $x_1 < x_2 < \ldots < x_{2k}$, $x_i \in L_m$ for $i = 1, \ldots, 2p$. Then the probability of a family of lattice paths to intersect the vertical line $x = (-R+m)\sqrt{3}/2$ at ordinates $x_1, \ldots, x_{2k}$ is equal to*

$$\tilde{P}_m(x_1, x_2, \ldots, x_{2k}) = \frac{1}{\tilde{Z}_m} \prod_{i=1}^{2p} \tilde{w}(x_i) \prod_{1 \leq i < j \leq 2p} (x_j - x_i)^2. \tag{6.1}$$

2. *Let $x_1 < x_2 < \ldots < x_k$, $x_i \in L_m^+$ for $i = 1, \ldots, k$. Then the probability of a family of p lattice paths not touching the x-axis to intersect the vertical line $x = (-R+m)\sqrt{3}/2$ at ordinates $x_1, \ldots, x_k$ is equal to*

$$\tilde{P}_m^{\mathrm{sym}}(x_1, x_2, \ldots, x_p) = \frac{1}{\tilde{Z}_m^{\mathrm{sym}}} \prod_{i=1}^{p} x_i^2 \tilde{w}(x_i) \prod_{1 \leq i < j \leq p} \left(x_j^2 - x_i^2\right)^2. \tag{6.2}$$

In both cases the weight function $\tilde{w} = \tilde{w}_m$ *is* even *and it is equal to*

$$\tilde{w}(z) = \frac{1}{(m/2 + k - 1/2 \pm z)!(R - m/2 + k - 1/2 \pm z)!}. \tag{6.3}$$

$\tilde{Z}_m$ *and* $\tilde{Z}_m^{\mathrm{sym}}$ *are normalisation constants.*

Note that every point in $\{(-R+m)\sqrt{3}/2\} \times L_m$ where none of the paths intersect is a centre point of a vertical tile. So the dual ensemble of (6.1) describes the distribution of vertical tiles along the line $x = (-R+m)\sqrt{3}/2$.

The weight function $\tilde{w}$ turns out to belong to the *Associated Hahn* family.

6.1 The Hahn and Associated Hahn ensemble

Let $[a,b] = [-1/2, 1/2]$ and $X_N = \{x_{N,n} = -1/2 + (2n+1)/2N,\ n = 0, \ldots, N-1\}$. The discrete orthogonal polynomial ensemble with weight function

$$w_N^{\mathrm{AHE}}(x_{N,n}; P, Q) = \frac{1}{n!(P-1+n)!(N-1+n)!(Q-1+N-1-n)!}$$

is called *Associated Hahn Ensemble* (AHE) [1, 6] with parameters P and Q. Its dual ensemble is known as *Hahn Ensemble* (HE) with weight $w_N^{\mathrm{HE}}(\,\cdot\,; P, Q)$. In [1, Sect. 2.4.2] the equilibrium measure $\mu^c_{\min}$ for the family $w_N^{\mathrm{HE}}(\,\cdot\,; AN+1, BN+1)$, $A, B > 0$, $c \in (0,1)$ fixed is computed. It turns out that there is exactly one band interval. For $A = B$ it is an interval $(-\beta, \beta)$ enclosed by two gaps of the same type. If $c < c_A = \sqrt{A^2 + A} - A$ those two gaps are voids for HE (and hence saturated regions for AHE with c substituted by $1-c$) and if $c > c_A$ they are saturated regions for HE (voids for AHE, c substituted by $1-c$). For the right band end point one has

$$\beta = \frac{\sqrt{c(1-c)(2A+c)(2A+c+1)}}{2(A+c)}. \tag{6.4}$$

The case $c = c_A$ is exceptional, leading to gaps consisting of single points.

6.2 The parameters in the $(2k, R, R)$-hexagon

With the change of variables $z = n - (m/2 + k - 1/2)$ and the substitutions $2k + m = N$ and $R - m + 1 = P$ in the weight function $\tilde{w}(z)$ in (6.3), we see that $\tilde{w}$ is from the AHE family with the parameters P and Q both equal to $R - m + 1$. With view on the half-hexagon and the general results of the previous sections we consider the parameter m in the distributions in Prop. 6.1 to be even, $m = 2p$, since this implies that the set L_{2p} has an even number $2N = 2k + 2p$ of elements. For $m = 2p + 1$, we have $0 \in L_m$ and the remarks at the end of Section 2 allow us to treat that case in the same fashion as below.

We choose R such that $R/k \to \lambda$ as $k \to \infty$. We scale the $(2k, R, R)$-hexagon by $1/k$, such that in the limit its intersection with he x-axis is the interval $[-\sqrt{3}\lambda/2, \sqrt{3}\lambda/2]$. We

want to study the intersection behaviour of the rescaled family of $2k$ lattice paths with a line $x = \tau \in (-\sqrt{3}\lambda/2, 0)$, i.e. the parameter $m = 2p$ shall satisfy the asymptotic relation $(-R+2p)\sqrt{3}/2k \to \tau$ for $k \to \infty$ and hence

$$2p/k \to 2\tau/\sqrt{3} + \lambda.$$

This implies for the ratio c of the numbers of paths ("particles") $2k$ and nodes $2N$

$$2k/2N = 2k/(2k+2p) \to c := \frac{2}{2 + \lambda + 2\tau/\sqrt{3}\lambda/2}.$$

Moreover for the parameters $P = R - 2p + 1$ and A we have

$$(P-1)/2k = 2NA/2k \to \lambda + \tau/\sqrt{3}$$

and hence we can choose

$$A = (-\tau/\sqrt{3})c.$$

The set of nodes $X_{2N} = \{l/k,\ l \in L_{2p}\}$ lies for large k in the interior of the intervall $[-1/c, 1/c]$. We can compute the band endpoints according to (6.4). Bearing in mind that that we are dealing with the dual ensemble of the Hahn ensemble, we have to substitute c by $1-c$. After a proper rescaling by $2/c$ the band endpoints read

$$\pm\frac{2}{c}\beta = \pm\sqrt{\lambda+1}\sqrt{1-\left(\frac{2}{\sqrt{3}\lambda}\tau\right)^2},\ \tau \in \left[-\frac{\sqrt{3}\lambda}{2}, \frac{\sqrt{3}\lambda}{2}\right]. \tag{6.5}$$

The points with these ordinates and abscissa τ lie precisely on the boundary of the inscribed ellipse of the rescaled hexagon. The neighbouring gap is a saturated region if $\tau < \tau_0 := -\frac{\sqrt{3}l^2}{2(2+l)}$, i.e. $1 - c > c_A$, cf. 6.1. Theorem 4.2 for the one-point correlation function yields that above the inscribed ellipse the line $x = \tau$ is fully packed with intersection points (and hence with the blue "up-tiles"), with probability exponentially close to one. If $-\frac{\sqrt{3}l^2}{2(2+l)} < \tau \le 0$, i.e. $1 - c < c_A$, then the gaps are voids and the part of the line $x = \tau$ above the ellipse is devoid of intersection points and hence fully packed with vertical (red) tiles with probability exponentially close to one. Close to a point with ordinate > 0 in the interior of the ellipse, by Theorem 4.1 the one-point correlation function takes a value strictly between 0 and 1 and hence both intersection points and vertical tiles occur with positive probability. This reflects the "arctic phenomenon" present in the large tiling of the half-hexagon depicted in Figure 2, namely the tiling being highly ordered at the corners and unordered in the interior, with a sharp transition along the inscribed ellipse. Furthermore, the fluctuations of the highest vertical tile ($\tau < \tau_0$) resp. the highest intersection point ($\tau > \tau_0$) are governed by the Tracy-Widom distribution by Theorem 4.3. Close to the x-axis approximations of the correlation functions for the intersection points of the paths with the line $x = \tau$ are also given by Theorem 4.1 in terms of the kernel S^0. This is in good accordance with a result in [13] on non-intersecting Brownian excursions, the continuous counterparts of our path model corresponding to tilings of the half-hexagon.

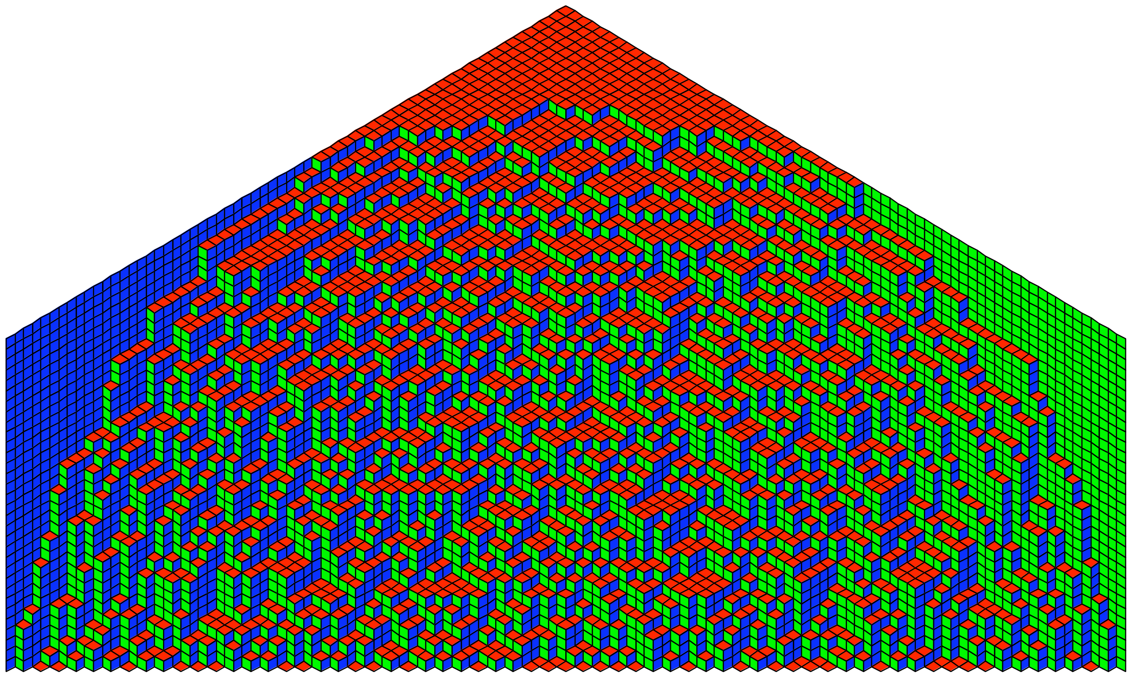

Figure 2: Arctic phenomenon in a $(32, 64)$-half-hexagon

7 Conclusion

We have applied the results of the monograph [1] to a general discrete ensemble a special case of which, namely random tilings of the half-hexagon, is applied in the study of minors of certain random matrices [4]. Our results on the correlations of tiles on a vertical line and the "arctic-half-ellipse" result complement the work in on the half-hexagon model.

The continuous counterpart of our general ensemble appears in the study of non-intersecting Brownian excursions [13] which matches the lattice paths formulation of the half-hexagon model. In [13] also the distribution area below the lowest path is addressed and the first moment thereof is computed. From a combinatorial point of view it would be interesting to (asymptotically) compute the expected area below the lowest path in a family of lattice paths. Furthermore, higher moments or the limiting distribution of area would be interesting if, say the number of paths is fixed and their length tends to infinity. This has so far only been fully accomplished for a single path in [11].

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